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TRANSPIRATION COOLING OF TANGENTIAL NEWTONIAN FLOW IN ANNULI: ANALYTICAL SOLUTIONS FOR TEMPERATURE DISTRIBUTIONS

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NOMENCLATURE

C_p, specific heat of fluid;
J_m, Bessel function of the first kind (*m*th order);
k, ratio of radius of inner cylinder to that of outer cylinder;
$\bar{k}$, thermal conductivity of fluid;
r, radial co-ordinate;
R_0, radius of outer cylinder;
t, time;
U, variable defined by equation (14);
v, local velocity;
Y_m, Bessel function of the second kind (*m*th order);
$Z_m(\lambda_m\xi)$, quantity defined by equation (6).

Greek symbols
ρ, fluid density;
μ, coefficient of shear viscosity;
ν, kinematic viscosity (μ/ρ);
λ, separation constant;
$\bar{\alpha}_1, \bar{\alpha}_2, \bar{\alpha}_3$, variables defined by equations (11)–(13) respectively;
Γ_1, Γ_2, constants defined by equations (9) and (10).

Subscripts
θ, angular component;
s, steady state solution;
v, solution in the presence of viscous dissipation of heat;
w, condition at the walls;
∞, solution at time $t = \infty$.

INTRODUCTION

OVER the past few years, transpiration cooling has been theoretically analyzed by several investigators in various geometries [1–3]. The theoretical knowledge of this method of cooling is of significant importance in space industry, combustion chamber walls, exhaust nozzles and porous walled reactors. The purpose of this note is to present analytical solutions for temperature distributions in steady as well as unsteady state heat transfer in laminar tangential flow between two infinite, coaxial cylinders in the presence of radial flow (see Fig. 1). The solutions are obtained in the absence as well as presence of viscous dissipation of heat. The results should be useful in evaluating the effectiveness of transpiration cooling of a wide variety of fluids in the Couette flow system shown in Fig. 1.

THEORETICAL

For the system illustrated by Fig. 1, at time $t < 0$, both inner and outer cylinders are maintained at temperature T_i. At $t > 0$, the temperatures of the inner and outer cylinders are changed to T_k and T_0 respectively. The incoming and outgoing fluids are assumed to be in equilibrium with the temperature at the two respective walls.

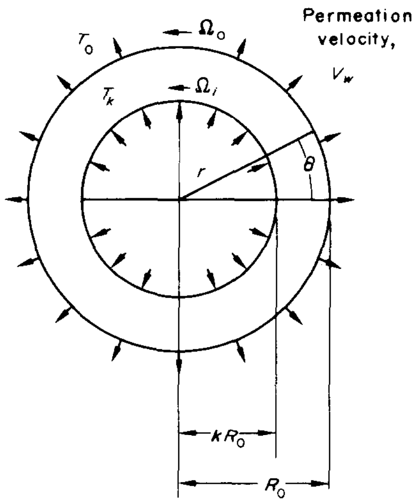

FIG. 1. Schematic of a porous annulus in which transpiration cooling is taking place.

For an incompressible fluid, the steady state tangential velocity distribution in the present system has been obtained by Shah and Au Yong [4]. The final result is

$$\phi = \left[\frac{-k\alpha-(1-\alpha)k^{1+p}}{k^{-1}-k^{1+p}}\right]\frac{1}{\xi} + \left[\frac{k\alpha+(1-\alpha)k^{-1}}{k^{-1}-k^{1+p}}\right]\xi^{1+p} \quad (1)$$

wherein,

$$\xi = r/R_0, \quad \phi = v_\theta/R_0\Delta\Omega, \quad -\alpha = \Omega_i/\Delta\Omega,$$
$$p = R_0 v_w/v, \quad \Delta\Omega = \Omega_0 - \Omega_i. \quad (2)$$

Neglecting the rate of work done on fluid by pressure forces and assuming that a step change in the wall temperature causes change in only thermal energy of the system, the energy balance equation in the dimensionless form can be written as (1)

$$Pr\frac{\partial \Lambda}{\partial \tau} = \frac{\partial^2 \Lambda}{\partial \xi^2} + \frac{(1-pe)}{\xi}\frac{\partial \Lambda}{\partial \xi} + Br_\theta\left\{\left[\frac{\mathrm{kp}\alpha + (1-\alpha)pk^{-1}}{k^{-1}-k^{1+p}}\right]\xi^p - \left[\frac{2k\alpha + 2(1-\alpha)k^{1+p}}{k^{1+p}-k^{-1}}\right]\frac{1}{\xi^2}\right\}^2 + \frac{4Br_r}{\xi^4} \quad (3)$$

wherein,

$$pr = \mu c_p/\bar{k}, \quad pe = p\,pr, \quad \Lambda = \frac{T-T_i}{T_k - T_0},$$
$$Br_\theta = \frac{pr}{C_p}\frac{R_0{}^2\Delta\Omega^2}{(T_k - T_0)}, \quad Br_r = \frac{pe}{C_p}\frac{pv^2}{R_0^2(T_k - T_0)},$$
$$\tau = \frac{\mu t}{\rho R_0^2}. \quad (4)$$

Equation (3) neglects the temperature variations in angular and axial positions. The terms involving Br_θ and Br_r take into account the viscous dissipation of heat due to angular and radial components of the velocity respectively.

Case 1: *No viscous dissipation of heat* ($\mathrm{Br}_\theta = \mathrm{Br}_r = 0$)

Under no viscous dissipation and with the boundary and initial conditions: at $\xi = k$, $\Lambda = \beta$; at $\xi = 1$, $\Lambda = -(1-\beta)$; and at $\tau = 0$, $\Lambda = 0$ where

$$\beta = (T_k - T_i)/(T_k - T_0)$$

the solution of equation (3) can be obtained by the standard method of separation of variables in a manner similar to the one described by Bird and Curtiss [5]. The final result for $\Lambda(\xi, \tau)$ when $pe/2$ is an integer can be shown to be

$$\Lambda(\xi,\tau) = \beta - \frac{k^{pe}}{k^{pe}-1} + \frac{1}{k^{pe}-1}\,\xi^{pe} - \sum_{m=1}^{\infty}\frac{\left\{\left[\frac{k^{pe}}{k^{pe}-1} - \beta\right]\left[Z_{(pe/2)-1}(\lambda_m) - k^{1-pe/2}Z_{pe/2-1}(\lambda_m k)\right] - \left[\frac{1}{k^{pe}-1}\right]\left[Z_{(pe/2)-1}(\lambda_m) - k^{1+(pe/2)}Z_{(pe/2)-1}(\lambda_m k)\right]\right\}}{\frac{\lambda_m}{2}\left[Z^2_{(pe/2)-1}(\lambda_m) - k^2 Z^2_{(pe/2)-1}(\lambda_m k)\right]}$$
$$\times\ \xi^{pe/2}[\exp(-\lambda_m^2\tau/pr)]\,Z_{(pe/2)}(\lambda_m\xi) \quad (5)$$

wherein

$$Z_{(pe/2)}(\lambda_m \xi) = J_{(pe/2)}(\lambda_m \xi) Y_{(pe/2)}(\lambda_m k) - J_{(pe/2)}(\lambda_m k) Y_{(pe/2)}(\lambda_m \xi) \quad (6)$$

and the eigenvalues λ_m are obtained from the solution of equation $Z_{(pe/2)}(\lambda_m \xi) = 0$ at $\xi = 1$.

The dimensionless temperature profiles for the values of pe and pr equal 4 and 1 respectively are plotted in Fig. 2 for $\beta = 0$ and $\beta = 1$. Temperature profiles for other values of β may be obtained by adding together the temperature profiles for $\beta = 0, 1$. The results shown in Fig. 2 indicate that for the values of pe equals 4 and pr equals 1, the steady state is virtually established when $\tau = 0{\cdot}2$. This time should be shortened with the increase in values of pr and pe.

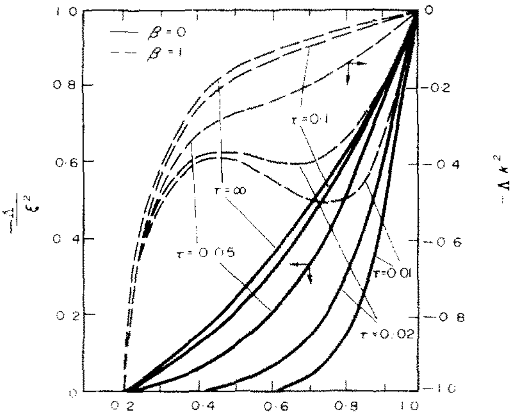

FIG. 2. Unsteady state temperature distribution in the absence of viscous dissipation of heat for $pr = 1$, $pe = 4$ and $k = 0{\cdot}2$.

For very small times, an alternate solution of $A(\xi, \tau)$ can be obtained by the Laplace transform method very similar to the one described by Bird and Curtiss [5].

Case 2: Presence of viscous dissipation of heat

The complete *steady state* solution of equation (3) in the presence of viscous dissipation of heat can be obtained by the standard method of variation of parameters [6]. The application of this method gives the result

$$A_s(\xi) = \beta + \frac{k^{pe}(1 + \Gamma_1) - \Gamma_2}{1 - k^{pe}} + \left[\frac{\Gamma_2 - 1 - \Gamma_1}{1 - k^{pe}}\right]\xi^{pe} + A_p(\xi) \quad (7)$$

wherein for $p \neq pe$ and $2p + 2 \neq pe$

$$A_p(\xi) = \frac{\bar{\alpha}_1 \xi^{2p+2}}{(2p+2)(2p+2-pe)} + \frac{\bar{\alpha}_2 \xi^p}{p(p-pe)} + \frac{\bar{\alpha}_3}{2\xi^2(pe+2)} \quad (8)$$

$$\Gamma_1 = \frac{\bar{\alpha}_1}{(2p+2)(2p+2-pe)} + \frac{\bar{\alpha}_2}{p(p-pe)} + \frac{\bar{\alpha}_3}{2(pe+2)} \quad (9)$$

$$\Gamma_2 = \frac{\bar{\alpha}_1 k^{2p+2}}{(2p+2)(2p+2-pe)} + \frac{\bar{\alpha}_2 k^p}{p(p-pe)} + \frac{\bar{\alpha}_3}{2k^2(pe+2)} \quad (10)$$

$$\bar{\alpha}_1 = -Br_\theta\left[\frac{kp\alpha + (1-\alpha)pk^{-1}}{k^{-1} - k^{1+p}}\right]^2 \quad (11)$$

$$\bar{\alpha}_2 = 2Br_\theta\left[\frac{kp\alpha + (1-\alpha)pk^{-1}}{k^{-1} - k^{1+p}}\right]\left[\frac{2k\alpha + 2(1-\alpha)k^{1+p}}{k^{1+p} - k^{-1}}\right] \quad (12)$$

$$\bar{\alpha}_3 = -4Br_r - Br_\theta\left[\frac{2k\alpha + 2(1-\alpha)k^{1+p}}{k^{1+p} - k^{-1}}\right]^2. \quad (13)$$

When $p = pe$ but $2p + 2 \neq pe$, the second terms on the right hand sides of equations (8)–(10) should be replaced by $\bar{\alpha}_2(\ln \xi\, \xi^{pe} - \xi^p/p)/pe$, $\bar{\alpha}_2/p\,pe$ and $(\bar{\alpha}_2/pe)\,(k^{pe}\ln k - k^p/p)$ respectively. When $p \neq pe$ but $2p + 2 = pe$, the first terms on the right hand sides of equations (8)–(10) should be replaced by $\bar{\alpha}_1(\ln \xi\, \xi^{pe} - \xi^{2p+2}/(2p+2))/pe$, $-\bar{\alpha}_1/pe(2p+2)$ and $(\bar{\alpha}_1/pe)\,[k^{pe}\ln k - k^{2p+2}/(2p+2)]$ respectively.

For the typical values of k, p, pe, α, Br_r and β, the effect of variation in the value of Br_θ on $A_s(\xi)$ is shown in Fig. 3. A

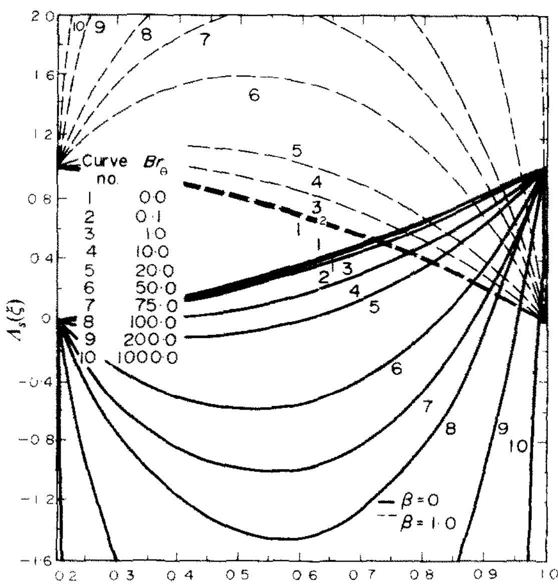

FIG. 3. Steady state temperature distribution at various values of Br_θ for $\alpha = 0{\cdot}5$, $Br_r = 0{\cdot}0$, $p = 1{\cdot}0$ and $pe = 2{\cdot}0$.

similar but more pronounced effect of the variation in the value of Br_r on $\Lambda_s(\xi)$ was observed.

The unsteady state dimensionless temperature distribution $\Lambda_v(\xi, \tau)$ in the presence of viscous dissipation of heat can be obtained in the straight forward manner by using the substitution

$$U(\xi, \tau) = \Lambda_v(\xi, \tau) - \Lambda_p(\xi) \quad (14)$$

wherein $\Lambda_p(\xi)$ is given by equation (8) and solving the resulting equation by the method of separation of variables. The final expression for $\Lambda_v(\xi, \tau)$ when $pe/2$ is an integer can be shown to be

$$\Lambda_v(\xi, \tau) = \Lambda_s(\xi) - \sum_{m=1}^{\infty} \frac{\int_k^1 \Lambda_s(\xi)\xi^{1-pe/2} Z_{pe/2}(\lambda_m \xi) d\xi}{\int_k^1 \xi [Z_{pe/2}^2(\lambda_m \xi)] d\xi} \times \xi^{pe/2}[\exp(-\lambda_m^2 \tau/pr)] Z_{(pe/2)}(\lambda_m \xi) \quad (15)$$

wherein $Z_{(pe/2)}(\lambda_m \xi)$ and eigenvalues λ_m are defined in the same manner as the ones in case 1.

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ASYMPTOTIC SOLUTIONS FOR FORCED CONVECTION FROM A ROTATING DISK

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Rotating disc systems are useful for precise heat and mass transport measurements. To analyze such measurements, an accurate expression for the Nusselt number $h\sqrt{(\nu/\omega)}/k$ (thermal) or $k_x\sqrt{(\nu/\omega)}/c\mathscr{D}_{AB}$ (binary) is required. The asymptotic formula of Levich [1],

$$Nu = 0{\cdot}620\, \Lambda^{\frac{1}{3}} \quad (1)$$

is suitable only when the Prandtl or Schmidt number, Λ, is very large: this formula overestimates Nu by 3 per cent even at $\Lambda = 1000$.

Gregory and Riddiford [2] have given a better expression,

$$Nu = \Lambda^{\frac{1}{3}}(1{\cdot}6126 + 0{\cdot}5705\, \Lambda^{-0{\cdot}36})^{-1} \quad (2)$$

for calculating Nu at moderately large Λ. This result is good within about 0·2 per cent for $\Lambda > 250$. Our purpose here is to give a more accurate result, valid down to $\Lambda \sim 1$, and also a new asymptote for $\Lambda \ll 1$. Our results are extensions of those given by Newman [7, 8] for $\Lambda \gg 1$ and $\Lambda \ll 1$, which came to our attention after this work was completed.

ANALYSIS

The formal solution for the Nusselt number on a rotating disc, in laminar flow with constant physical properties, is

$$Nu = \frac{1}{J(\Lambda)} \quad (3)$$